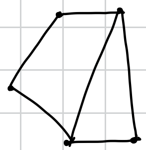
Graph Theory

1.1 Graphs

A graph is a pair $G = (V, E)$ where V is a set called vertex set and E is a set of unordered pairs in V. E is called the edge set.

e.g.



$V(G)$ = vertex set of G

$E(G)$ = edge set of G

We will write (u, v) for the edge $\{u, v\}$

$$\begin{array}{c}(u,v)\\ \parallel \\ (v,u)\end{array}$$

$|G| = |V(G)|$

$e(G) = |E(G)|$

Remark: sometimes we will have multiple edges between u and v

In that case, G is a multigraph

We will sometimes have loops which are edge (v, v)

A simple graph is one without loops and multiple edges.

Def. - $u, v \in V(G)$ are called <u>adjacent</u> if $(u, v) \in E(G)$

- An edge $e \in E(G)$ is <u>incident</u> to $v \in V(G)$ if $v \in e$

- Edges $e, e' \in E(G)$ are <u>incident</u> if $e \cap e' \neq \emptyset$

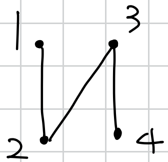
- If $(u, v) \in E(G)$, then v is a <u>neighbour</u> of u

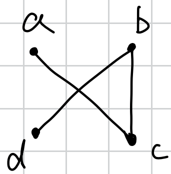
Examples

- V = set of people in room
  E = pairs of people who met the first time today

- V = set of cities in a country
  E = form connection

- V = users on Facebook
  E = friends





1.2 Graph isomormism 图同构

$V(G_1) \to V(G_2)$

$\phi : G_1 \to G_2$ is a graph isomorphism

if it is a bijection from $V(G_1)$ to $V(G_2)$

and $(u,v) \in E(G_1)$ iff $(\phi(u), \phi(v)) \in E(G_2)$

$\phi(1) = a \quad \phi(2) = c$
$\phi(3) = b \quad \phi(4) = d$

Isomorphism is an equivalence relation.

Unlabelled graph = isomorphism class
(equivalence of the isomorphism relation)

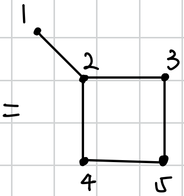
1.3 Adjency and incidence matrix

Let G be a graph with vertex set $[n] = \{1, 2, 3, \dots n\}$

The adjacency matrix $A(G)$ is an $n \times n$ matrix such that $A_{ij} = \begin{cases} 1, & \text{if } (i,j) \in E(G) \\ 0, & \text{otherwise} \end{cases}$

Note that $A_{ij} = A_{ji}$ so A is symmetric. and real

so A has real eigenvalues 特征值

$G =$ 

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Let $V(G) = \{v_1, v_2, \dots v_n\}$ and $E(G) = \{e_1, e_2, \cdots, e_m\}$

Then the incidence matrix $B(G)$ is $n \times m$ matrix such that

$$B_{ij} = \begin{cases} 1, & \text{if } v_i \in e_j \\ 0, & \text{otherwise} \end{cases}$$

Observation: every column of B has two entries that are equal to 1

1.4 Degree

Given a vertex $v \in V(G)$, we write $N(v)$ for the set of neighbours of v.

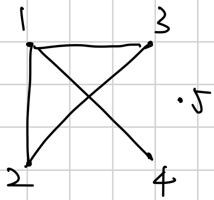
$N(v)$ is called the neighbourhood of v.

The degree $d(v) = |N(v)|$

A vertex is isolated if $d(v) = 0$

$d(v)$ is the number of 1 entries in the row corresponding to v in $A(G)$
$B(G)$

Example



$d(1)=3$
$d(2)=2$
$d(3)=2$
$d(4)=1$
$d(5)=0$

Fact $BB^T = D + A$ where $D = \begin{pmatrix} d(1) & \cdots & & 0 \\ \vdots & d(2) & & \vdots \\ & & \ddots & \\ 0 & \cdots & & d(n) \end{pmatrix}$

$$(BB^T)_{ij} = \sum_{k=1}^{m} B_{ik} B^T_{kj}$$

$$= \sum_{k=1}^{m} B_{ik} B_{jk} = \sum_{k=1}^{m} \mathbb{1}\{v_i \in e_k, v_j \in e_k\}$$

$$= \begin{cases} \mathbb{1}(i,j) \in E(G), & \text{if } i \neq j \\ d(i), & \text{if } i = j \end{cases}$$

The minimum degree of a graph G is the smallest $d(v)$ over all $v \in G$
maximum largest

$\delta(G)$ = minimum degree $\Delta(G)$ = maximum degree

The average degree of G is $\bar{d}(G) = \dfrac{\sum_{v \in V(G)} d(v)}{|V(G)|}$

A graph is d-regular if $d(v) = d \ \forall v \in V(G)$

Q: Is there a 3-regular graph on 9 vertices?

Lemma: $\sum_{v \in V(G)} d(v) = 2e(G)$

Proof: Each edge $(u,v) \in E(G)$ contributes 2 to the sum $\sum_{w} d(w)$: once $d(u)$ and once in $d(v)$.

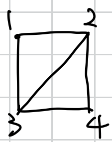
1.5 Subgraphs

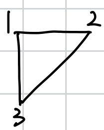
A graph $H=(U,F)$ is a subgraph of $G=(V,E)$ if $U \subseteq V$ and $F \subseteq E$

由于 graph 的定义，
F中仅含 U中的点组成的edge

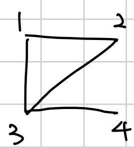
Induced subgraph: For each $U \subseteq V(G)$ the induced subgraph $G[U]$ is the graph with vertex set U and edges set $\{e \in E(G) : e \subseteq U\}$

Spanning subgraph: If $H=(U,F)$ is a subgraph of $G=(V,E)$ and $U=V$ then H is the spanning subgraph in G

G =




induced
but not spanning



spanning
not induced

1.6 Complete graph

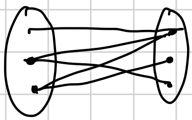
- K_n complete graph on n vertices

  We take all edges between n vertices

$$e(K_n) = \frac{n(n-1)}{2}$$

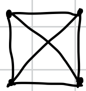
- E_n empty graph on vertices no edges.

- Biparte graph has two disjoint set of vertices U and V such that $\forall v \in U \cup V$ and every edge contains one vertex from U and one vertex from V

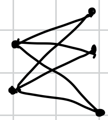


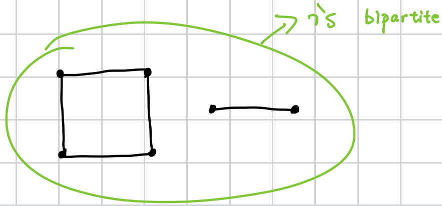
- Complete bipartite graph
  $K_{n,m}$ has all possible edges between a set of size n and a set size of m

Examples K_4



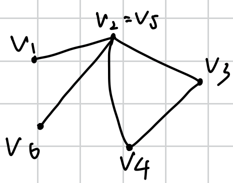
$K_{2,3}$





1.7 Walks, paths, cycles

Def. A sequence vertices $(v_1, v_2, \cdots, v_t)$ is a walk if $(v_i, v_{i+1}) \in E(G)$ $\forall$ $i \le t-1$



A walk $(v_1, v_2, \cdots, v_t)$ is a path if $v_1, \cdots, v_t$ are distinct

A cycle is a walk $(v_1, \cdots, v_t)$ such that $v_1 = v_t$ and $v_1, \cdots, v_{t-1}$ are distinct

The length of a walk is the number of edges (counted multiple times for edges used in multiple times) in the walk.

Proposition. Every walk from u to v contains a path from u to v

— Proof: By induction. on the length of the walk

If length $= 1$, it's correct

Take a walk $(v_1, v_2, \cdots, v_t)$ from u to v

Either this is a path or $\exists\, i<j$, such that $v_i = v_j$

By removing the vertices $v_{i+1}, v_{i+2}, \cdots, v_{j-1}$

and merge v_i and v_j

you got a shorter walk from u to v

Proposition 1.33

Every graph G with minimum degree $\delta \geq 2$ contains a path of length δ and a cycle of length at least $\delta + 1$

Proof.

Let $v_1, \cdots, v_k$ be a longest path in G. Then all the neighbours of v_k must belong to $\underbrace{v_1, \cdots, v_{k-1}}$

否则这条path可以更长

so we have $\underbrace{k-1 \geq \delta}$ $\Rightarrow$ $\underbrace{k \geq \delta + 1}$ → δ+1个点的path长为δ

有 k-1 个可能邻节点，度数不超过这个

由于 $\delta(V_k) \geq 2$

所以 V_k 的邻节点 $\begin{cases} V_{k-1} \\ V_1, \cdots V_{k-2} \end{cases}$

至少在 $V_1, \ldots V_{k-2}$ 中选一个

那么长度最小的环一定是从 V_k 开始往前找 $\delta(V_k)$ 个连续的点，构成一个长度为 $\delta+1$ 的环

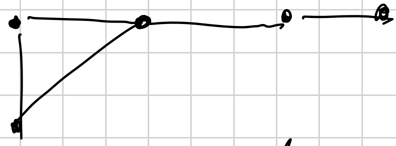
Remark

We have also proved that a graph with minimum degree $\delta \geq 2$ contains cycles of at least $\delta - 1$ different lengths. This fact, and the statement of Prop 1.33, are both tight; too see this, consider the complete graph $G = K_{\delta+1}$

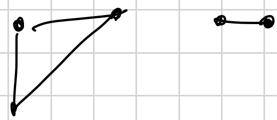
1.8 Connectivity 连通性

Definition: A graph G is connected if for all pairs $u, v \in G$, there is a path in G from u to v.

Note that it suffices for there to be a walk from u to v



connected



not connected

Def. A (connected) component of G is a connected subgraph which is maximal with respect to inclusion.

We say that G is connected iff it has exactly one component.

Proposition 1.39.

A graph with n vertices and m edges has at least $n-m$ connected components.

1.9 Graph operations and parameters

Def. Given $G=(V,E)$, the complement $\bar{G}$ of G is the graph on the same vertex set V and $(u,v)\in E(\bar{G})$ iff $(u,v)\notin E(G)$

补图定义

Def A clique in G is a complete subgraph in G

An independent set is an empty induced subgraph in G

Notation. Let $\omega(G)$ be the number of vertices in a clique of G of maximum size.

Let $\alpha(G)$ be the number of vertices in an independent set of G of maximum size.

Claim: A vertex subset $U \subseteq V(G)$ is a clique in G, iff $U \subseteq V(\overline{G})$ is an independent set in $\overline{G}$

Corollary: We have $\omega(G) = \alpha(\overline{G})$
and $\alpha(G) = \omega(\overline{G})$

2. Trees

2.1. Trees

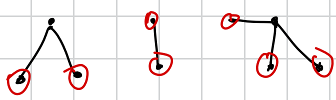
Def: A graph having no cycle is acyclic.

A forest is an acyclic graph.

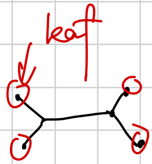
A tree is a connected acyclic graph

A leaf is a vertex of degree 1

Example



Forest



Forest / tree

Lemma 2.3.

Every finite tree with at least 2 vertices has at least two leaves

Deleting a leaf from an n-vertex tree produces a tree with n-1 vertices.

$u, w \in G'$, there's a path between u and w

if v in the path.

$d(v) \geq 2$

but v is a leaf, $d(v) \neq 1$

so v can't be in any path. for all $u, w \in G'$

so G' is still connected

2.2. Equivlant definition of trees

Theorem: For a n-vertex graph G ($n \geq 1$), the following are equivalent (and characterize the trees with n vertices)

(a) G is connected and has no cycles

(b) G is connected and has $n-1$ edges

(c) G has $n-1$ edges and no cycles

(d) For every pair $u, v \in V(G)$, there is exactly one u,v-path in G

Def An edge of a graph is a cut-edge if its delection disconnects the graph

Lemma An edge contained in a cycle is not a cut-edge

↳ proof: let (u,v) belong to a cycle

any path from x to y in G which uses edge (u,v) can be extended

to a walk in $G \setminus (u,v)$.

In particular, this is a walk in $G \setminus (u,v)$

对于 u, v 若有两条 path P, Q,

则 存在 $(x,y) \in P \setminus Q$

则 $P \cup Q \setminus \{x,y\}$ 是一条 x 到 y 的 walk

Def 2.7. Given a connected graph G, a spanning tree T is a subgraph of G which is a tree and contains every vertices of G.

Corollary 2.8

(a) every connected graph on n vertices has at least $n-1$ edges and contains a spanning tree

(b) Every edge of a tree is a cut-edge

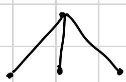
(c) Adding an edge to a tree creates exactly one cycle.

2.3. Cayley's formula

Question 2.9
How many spanning trees are there in an n-vertex labelled complete graph

Example
$n=3$:

$n=4$ degree 3



$\Rightarrow 4$

or every vertex has degree ≤ 2

$\Rightarrow \frac{4!}{2} = 12$

$\Rightarrow$ 16 in total

Theorem (Cayley's formula)

There are n^{n-2} labelled trees on n-vertices.

Proof (1)

We will construct a bijection between n-vertex labelled tree and sequences of length $n-2$ in which each elemnt is in $[n]$.

Numbers of such sequences is n^{n-2}

Definition (Prüfer code)

Let T be a tree whose vertex set is some ordered set S of size n.

for example, $[n]$

$f(T)$ is a sequence in S^{n-2}

We delete the leaf whose label is the smallest (among the leaves)

Call this vertex v.

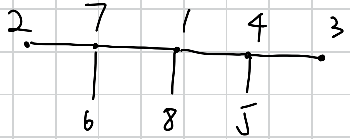
v has unique neighbour in T. The first element of $f(T)$ is defined as the label of this neighbour.

Then iterate this with $T-v$.

Eventually, when we have 2 vertices left in the tree, we stop.

This defines a sequence $f(T)$ in S^{n-2}

$T=$



7.	4	4	1	7	1
(-2)	(-3)	(-5)	(-4)	(-6)	(-7

Proposition:

The map $T \mapsto f(T)$ is a bijection between trees labelled S and S^{n-2}

Proof:

By induction on n

$n=2$ Trivial ✓

$n>2$: How to get a tree from a sequence $(a_1, a_2, \cdots, a_{n-2}) \in S^{n-2}$

Claim: If $S(T) = (a_1, a_2, \cdots, a_{n-2})$ then. $\{a_1, \cdots, a_{n-2}\}$ is the set of non-leaf vertices in T

Proof of claim: if v is a leaf, then we cannot remove the unique neighbour of v before v (as they would disconnect the graph).

Hence, v will not appear in the sequence

If v is not a leaf, then it has at least two neighbours in T.

Before we remove v, we must remove at least one neighbour of v.

At that point, you add the label of v to the sequence

The leaf of smallest label is the minimal element in the set $S \setminus \{a_1, \dots, a_{n-2}\}$.

by claim, Hence if $f(T) = (a_1, \dots, a_{n-2})$

Then, the minimal leaf of T is the minimal element in $S \setminus \{a_1, \dots, a_{n-2}\}$, Call this $v \in S$

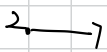
Then, let $T' = T - v$ Then, $f(T-v) = (a_2, \dots, a_{n-2})$

There is an unique T' (by induction) with $f(T') = (a_2, \dots, a_{n-2})$

T must be formed by attaching the edge (v, a_1) to T'.

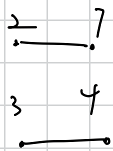
$f = (7, 4, 4, 1, 7, 1)$

$v = 2$,



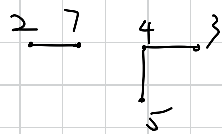
$f = (4, 4, 1, 7, 1)$

$v = 3$



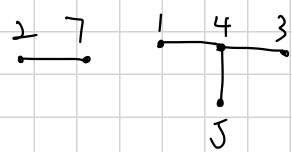
$f(4, 1, 7, 1)$

$v = 5$



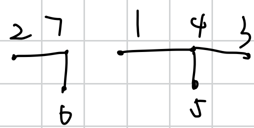
$f(1, 7, 1)$

$v = 4$



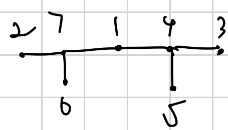
$f(7, 1)$

$v = 6$



$f(1)$

$v = 7$



Recall

Def: A directed graph (digraph) consists of a vertex set and an edge set of ordered pairs of vertices.

In-degree / Out-degree 入度/出度

To each function $f: [n] \to [n]$,

I will assign a tree with two special vertices L and R

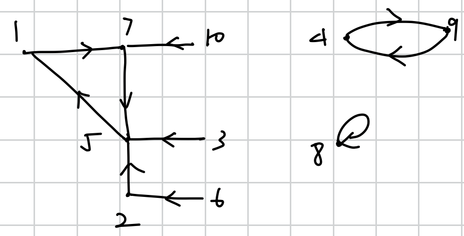
Given $f: [n] \to [n]$

I want to define a digraph G_f on vertex set $[n]$

by taking a directed edge $(i, f(i))$ for each $i \in [n]$

Example: $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 5 & 5 & 9 & 1 & 2 & 5 & 8 & 4 & 7 \end{pmatrix}$

$G_f =$



Every connected component has as many edges as vertices.
(each vertex has out-degree 1)

Each component has a unique cycle.

Delete all cycles

We put a path on the vertices of these cycles.

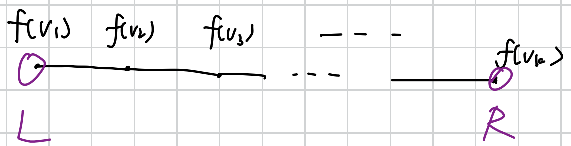
Let M be the set of vertices in the cycle.

f defines a bijection from M to itself

$$f|_M = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \\ f(v_1) & f(v_2) & \cdots & f(v_k) \end{pmatrix} \quad v_1 < v_2 < \cdots < v_k$$

$f(v_1), f(v_2), \cdots, f(v_k)$ is a permutation of $v_1, \cdots, v_k$.

Put a path



Connetivity

Def. In a connected graph G, a set $S \subseteq V(G)$ is a vertex cut (or cut) if $G \setminus S$ is disconnected.

Here $G \setminus S = G[V(G) \setminus S]$

If $\{v\}$ is a vertex cut, then we say v is a cut vertex.

Def. A graph G is k-connected if $|V(G)| > k$ and if S is a vertex set, then $|S| \geq k$

(i.e., for every $X \subseteq V(G)$ of size at most $k-1$, $G \setminus X$ is connected)

The contivity of G, denoted as $\kappa(G)$ is the largest k such that G is k-connected.

Example: $\kappa(K_n) = n-1$ n点完全图是 n-1 联通的

$\kappa(K_{n,m}) = \min(n, m)$ 完全二分图 ⇒ 移除数量较小的一侧

1-connected $\Leftrightarrow$ connected.

only for G with $|V(G)| > 1$

Proposition For every graph G, $\kappa(G) \leq \delta(G)$

Proof: We need to prove that either

we can remove at most $\delta(G)$ vertices to make the graph disconnected

or $|G| \leq \delta(G) + 1$

Let v be a vertex of degree $\delta(G)$

Let $S = N(v)$ Now $G \setminus S$ is not connected.

unless there are no vertices in G outside of $\{v\} \cup N(v)$

In the better case, $|G| \le 1 + \delta(G)$

Remark Large minimum degree does not imply large connetivity.

For example. two disjoint copies of K_n

Theorem (Mader 1972) Every graph of average degree at least $4k$ has a k-connected subgraph.

proof

3.2. edge connectivity

Def. A disconnecting set of edges is some $F \subseteq E(G)$ such that. $G \setminus F$ is not connected

Given $S, T \subset V(G)$, we write $[S, T]$ for the set of edges with one endpoint in S and the other in T

An edge cut is a set of edges of the form $[S, \bar{S}]$ for some non-empty and proper $S \subset V(G)$

Remark Every edge-cut is disconnecting set.

Not every disconnecting set is an edge cut

But every minimal disconnecting set is an edge cut.

A graph is k-edge-connected if every disconnecting set has size at least k.

The edge connectivity of G denoted $\kappa'(G)$ is the longest k such that. G is k-edge connected (Equ , minimum size of a disconnecting set is $\kappa'(G)$).

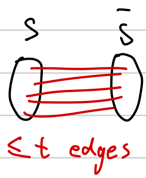
A. disconnecting set of size 1 is called a bridge.

Theorem For every graph G, we have $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

proof $\kappa'(G) \leq \delta(G)$

You can remove all the edges incedent to $\delta(G)$ to make the graph disconnected.

$\kappa(G) \leq \kappa'(G)$.

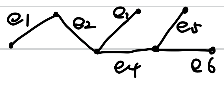


x

Recall For a graph G, the line graph $L(G)$ has vertex set $E(G)$ with e and f adjacent in $L(G)$ if they share a vertex in $V(G)$

What is a path in $L(G)$?

$e_1, e_2, \cdots, e_k \in E(G)$ s.t. $e_i \cap e_{i+1} \neq \emptyset$



If $e_{i-1} \cap e_i \cap e_{i+1} \neq \emptyset$ then delete e_i

You end up with a path from e_1 to e_k.
in G

Corollary Let u and v be vertices in G

(i) If $(u,v) \notin E$, then the min number of vertices distinct from u and v separating from u to v is equal to the max number of internally vertex-disjoint u-v paths in G.

若 $(u,v) \notin E$, u到v的 min 内部顶点割大小 = max u到v 顶点不交路径.数量.

删除这些点, u和v断开

proof: $S = N(u)$ $T = N(v)$
apply Menger's Theorem.

(ii) The min number of edges separating u from v in G is equal to the max number of edge-disjoint paths between u and v

从u到v的 min 边割 = max u到v 顶点不交路径.数量.

Theorem Global Menger

(a) A graph is k-connected iff it contains k internally vertex-disjoint paths between any two vertices (and has at least 2 vertices).

(b) A graph is k-edge-connected iff between any distinct u, v there are k edge-disjoint paths.

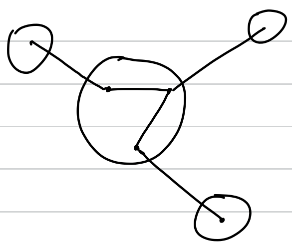
(c) If there are k internall

Def A trail is a walk with no repeated edges.

Def An Eulerion trail in a (multi)graph G is a walk in G possing through every edge exactly once. If this walk is closed, it is called an Eulerion tour.

Def A connected (multi)graph has an Eulerian tour iff each vertex has even degree

Lemma Every maximal trail is an even multigraph. is a closed trai



Proposion If G is Hamiltonian, then for any non-empty $S \subset V(G)$

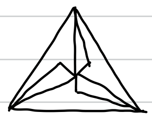
$G \setminus S$ has at most $|S|$ connected components

Collery If a (connected) Hamiltonian bipartite graph has bipartation A and B, then $|A| = |B|$

Proof Let $S = A$ Then $G \setminus S$ is independent set (empty graph) of size of $|B|$

By proposition, $|B| \leq |S| = |A| \Rightarrow$ By symmetry $|A| \leq B \Rightarrow |A| = |B|$

Remark The condition in the proposition is not sufficient.

$G=$ 

Theorem If G is an n-vertex graph with $\delta(G) \geq \frac{n}{2}$ and $n \geq 3$, then G is Hamiltonian.